2.1. Levi-Civita connection on a submanifold.

Let $(\bar{M}, \bar{g})$ be a Riemannian manifold with Levi-Civita connection $\bar{D}$, and let M be a submanifold of $\bar{M}$, equipped with the induced metric $g := i^*\bar{g}$, where $i\colon M \to \bar{M}$ is the inclusion map. Show that the Levi-Civita connection D of (M, g) satisfies $D_X Y = (\bar{D}_X Y)^{\mathrm{T}}$ for all $X, Y \in \Gamma(TM)$, where the superscript T denotes the component tangential to M and $\bar{D}_X Y$ is defined(!) as $\bar{D}_X Y := \bar{D}_{\bar{X}} \bar{Y}$ for any extensions $\bar{X}, \bar{Y} \in \Gamma(T\bar{M})$ of X, Y.

2.2. Gradient and Hessian form. Let (M, g) be a Riemannian manifold, D the Levi-Civita connection and $f\colon M \to \mathbb{R}$ a smooth function on M.

(a) The *gradient* $\operatorname{grad} f \in \Gamma(TM)$ is defined by

$$df(X) = g(\operatorname{grad} f, X), \quad \forall X \in \Gamma(TM).$$

Compute $\operatorname{grad} f$ in local coordinates.

(b) The *Hessian form* $\operatorname{Hess}(f) \in \Gamma(T_{0,2}M)$ is defined by

$$\operatorname{Hess}(X, Y) = g(D_X \operatorname{grad} f, Y), \quad \forall X, Y \in \Gamma(TM).$$

Prove that $\operatorname{Hess}(f)$ is symmetric and compute $\operatorname{Hess}(f)$ in local coordinates.

2.3. The exponential map for $\mathrm{SO}(n, \mathbb{R})$. We consider the matrix group

$$G := \mathrm{SO}(n, \mathbb{R}) = \{g \in \mathbb{R}^{n\times n} : g^{-1} = g^{\mathrm{T}},\ \det(g) = 1\}$$

which acts on $\mathbb{R}^{n\times n}$ by matrix multiplication. Recall that G is a manifold with tangent bundle

$$TG = \{(g, gA) : g \in G,\ A \in \mathbb{R}^{n\times n},\ A^{\mathrm{T}} = -A\}.$$

We say that $X \in \Gamma(TG)$ is a *left-invariant vector field* on G if there is some $X_0 \in TG_e$ such that $X_g = gX_0$ for all $g \in G$.

(a) A Riemannian metric $\langle \cdot, \cdot \rangle$ on TG is called *bi-invariant* if both, left translation $l_g\colon G \to G$, $l_g(x) = gx$ and right tanslation $r_g\colon G \to G$, $r_g(x) = xg$ are isometries.

Prove that

$$\langle (g, gA), (g, gB) \rangle := \tfrac{1}{2}\operatorname{trace}(AB^{\mathrm{T}})$$

defines a bi-invariant Riemannian metric on G.

(b) Let D be the Levi-Civita connection with respect to $\langle \cdot, \cdot \rangle$ on G. Show that for left-invariant vector fields $X, Y \in \Gamma(TG)$ we have

$$D_X(Y) = \tfrac{1}{2}[X, Y].$$

Hint: Use that the Lie-bracket of left invariant vector fields is left-invariant and satisfies $[A, B] = AB - BA$ on TG_e.

(c) Prove that the exponential map $\exp_e \colon TG_e \to G$ is given by

$$\exp_e(A) = e^A := \sum_{k=0}^{\infty} \frac{A^k}{k!}$$